# Recursion Relations for Five-Point Conformal Blocks and Beyond: A Practical Approach

Valentina Prilepina

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Institute for Theoretical and Mathematical Physics (ITMP)

based on

**arXiv:2103.12092 [hep-ph]**

with David Poland and ongoing work with Petar Tadic

# Why Study Conformal Field Theories (CFTs)?

CFTs describe universal physics of scale invariant critical points:

- continuous phase transitions in condensed matter and statistical physics systems
- fixed points of RG flows

Provide a handle on

- Universal structure of the landscape of QFTs
- Quantum gravity via the AdS/CFT correspondence and holography
- String theory
- Black holes

# The Conformal Bootstrap

Conformal bootstrap program seeks to systematically apply

- conformal symmetry
- crossing symmetry
- unitarity/reflection positivity

to map out and solve the space of allowed CFTs

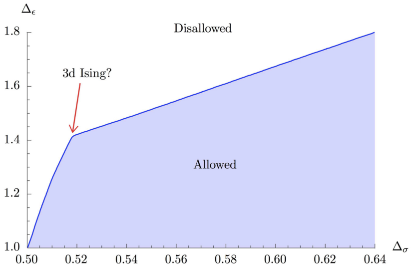



Figure: Upper bound on $\Delta_\epsilon$ as a function of $\Delta_\sigma$ in 3d CFTs [El-Showk, Paulos, Poland, Rychkov, Simmons-Duffin, Vichi, '12; '14]

# The Ultimate Dream

- Owing to bootstrap: tremendous progress on the numerical and analytic fronts! e.g. Ferrara et al. (1971, 1973), Dobrev et al. (1976, 1977), Polyakov (1974), Dolan & Osborn (2001, 2004, 2011), Poland et al. (2012), Simmons-Duffin (2014), El-Showk et al. (2014), Kos et al. (2014, 2015, 2016), Costa & Hansen (2015), Rejon-Barrera & Robbins (2016), Echeverri et al. (2016), Costa et al. (2016), Fortin & Skiba (2016, 2019), Karateev et al. (2017), Poland & Simmons-Duffin (2019)
- Dream: to classify and solve the entire landscape of CFTs and predict their observables

CFTs are signposts in the landscape of QFTs!

# Outline

I. Five-Point Functions

- Motivation
- What is Known
- Form

II. Weight-Shifting Operator (WS) Formalism

- Weight-Shifting Operators
- Crossing Relations
- Gluing 3-point Functions to Form Conformal Blocks

# Outline (cont.)

III. Recursion Relations from Weight-Shifting Operators

- Recovering Known Recursion Relation for 4-point Blocks
- Derivation of 5-point Recursion Relations
- Main Results
- Discussion

# Outline (cont.)

IV. Promoting $\Phi$ to a Spinning Operator

- Spin 1 Case
- Comment on Spin 2 Case

V. The Averaged Null Energy Condition (ANEC): An Application

- Discussion of Possible Constraints

# Outline (cont.)

VI. Ongoing Work: Moving Beyond 5-point Blocks

- Generalization to the 6-point Snowflake Channel

VII. Ongoing Work: The 5-point Conformal Bootstrap

- Conformal Bootstrap for the 3D Ising Model via 5-point Blocks

VIII. Conclusions

# Motivation for Studying Higher-Point Functions

So far, most results extracted by considering 4-point functions!

(for a review, see e.g. Poland, Rychkov and Vichi (2019))

$\Rightarrow$ Explicit expressions or recursion relations for conformal blocks appearing in 4-point functions of scalars in arbitrary $d$

$\Rightarrow$ Rich variety of techniques for handling 4-point blocks in arbitrary Lorentz representations

# Motivation for Studying Higher-Point Functions (cont.)

Many reasons to desire a precise understanding of 5- and higher-point functions!

1. Multipoint bootstrap Rosenhaus (2018), Parikh (2019), Bercini et al. (2021), Antunes et al. (2021)
2. Better access to different physical regimes of a CFT
3. New probe into $\langle \mathcal{O}_H \mathcal{O}_H \mathcal{O}_H \rangle$ in holographic CFTs via 5-point object $\langle \mathcal{O}_L \mathcal{O}_L \mathcal{O}_H \mathcal{O}_L \mathcal{O}_L \rangle$
4. Improved understanding of CFT implications of the ANEC

# What is Known So Far

A few key developments include

- Five-point scalar exchange conformal blocks first computed by Rosenhaus (2018)
- Holographic representations of higher-point conformal blocks constructed by Parikh (2019, 2020) and Hoback and Parikh (2021)
- Dimensional reduction formulae for higher-point scalar exchange blocks derived by Hoback and Parikh (2020)
- General representations of higher-point scalar exchange blocks developed by Fortin, Ma, Skiba (2019, 2020) using the operator product expansion (OPE) in embedding space

# What is Known So Far (cont.)

- Few explicit results for higher-point conformal blocks capturing exchange of spinning operators exist
- Exception: Series expansion for general 5-point blocks with identical external scalars developed by Gonçalves et al. (2019)
- Lightcone blocks for five- and six-point functions in the snowflake channel obtained by Antunes et al. (2021)
- Multipoint comb channel blocks obtained in 3D and 4D via a connection to Gaudin integrable models by Buric et al. (2021)

# Goal of this Work

Here we seek to

- Identify a simple and practical approach to computing 5-point blocks
- Improve and extend our understanding of 5-point blocks by deriving simple recursion relations

We

- Consider scalar 5-point function $\langle \phi_{\Delta_1} \phi_{\Delta_2} \Phi_{\Delta_3} \phi_{\Delta_4} \phi_{\Delta_5} \rangle$
- Compute the conformal block for arbitrary symmetric traceless tensor exchange in (12) and (45) OPEs

Our results

- May be seen as a natural generalization of recursion relations for 4-point blocks obtained by Dolan & Osborn (2011)

# Setting the Stage: 5-point Functions

- Work in the index-free embedding formalism of Costa et al. (2011)
- Restrict to parity-even correlators only
- Label spin-$\ell$ primaries by $\chi \equiv [\Delta, \ell]$

Conformal invariance fixes 5-point function of spin-$\ell$ primaries to have the form

$$\langle \mathcal{O}_1(X_1; Z_1) \cdots \mathcal{O}_5(X_5; Z_5) \rangle = \prod_{i<j}^{5} X_{ij}^{-\alpha_{ij}} \sum_k f_k(u_a)\, Q^{(k)}_{\chi_1,\ldots,\chi_5}(\{X_i; Z_i\}),$$

where $X_{ij} = -2X_i \cdot X_j$ and

$$\alpha_{ij} = \frac{1}{3}\left(\tau_i + \tau_j - \frac{1}{4}\sum_{k=1}^{5} \tau_k\right)$$

with $\tau_i = \Delta_i + \ell_i$

# Setting the Stage: 5-point Functions (cont.)

In this form,

- Factors $X_{ij}$ carry powers fixed by homogeneity
- $f_k(u_a)$ is some function of the conformal cross-ratios $u_a$
- Polynomials $Q^{(k)}$ have weight $\ell_i$ in each point $X_i$, degree $\ell_i$ in each $Z_i$
- $Q^{(k)}$ must be identically transverse, i.e.

$$Q^{(k)}_{\chi_1,\dots,\chi_5}(\{\lambda_i X_i; \alpha_i Z_i + \beta_i X_i\}) = Q^{(k)}_{\chi_1,\dots,\chi_5}(\{X_i; Z_i\}) \prod_i (\lambda_i \alpha_i)^{\ell_i}$$

$Q^{(k)}$ constructed from basic building blocks

- $V_{i,jk}$
- $H_{ij}$

of the standard box tensor basis

# Setting the Stage: 5-point Conformal Blocks

May expand $\sum_k [\ldots]$ in a basis of conformal blocks, which

- Capture the exchange of specific primary operators in the OPE
- Are the building blocks of CFT correlation functions
- Effectively encode the kinematical contribution of descendant operators in terms of primary operators

Choose to compute blocks in double OPE channel (12)(45)

# Setting the Stage: 5-point Conformal Blocks (cont.)

Consider the scalar 5-point function

$$\langle \phi_{\Delta_1}(X_1)\phi_{\Delta_2}(X_2)\Phi_{\Delta_3}(X_3)\phi_{\Delta_4}(X_4)\phi_{\Delta_5}(X_5)\rangle$$

- Insert a projector $|\mathcal{O}_{\Delta,\ell}|$ onto the conformal multiplet of $\mathcal{O}_{\Delta,\ell}$ (similarly for $\mathcal{O}'_{\Delta',\ell'}$) into the 5-point function

$$|\mathcal{O}| \equiv \frac{1}{\mathcal{N}_{\mathcal{O}}}\int D^d X |\mathcal{O}(X)\rangle\langle\tilde{\mathcal{O}}(X)|$$

- Each 3-point function $\langle \mathcal{O}_{\Delta,\ell}\Phi_{\Delta_3}\mathcal{O}'_{\Delta',\ell'}\rangle$ expanded in a basis of tensor structures
- Tensor structures labeled by index $a$
- Each comes with an independent coefficient $\lambda^a_{\mathcal{O}_{\Delta,\ell}\Phi_{\Delta_3}\mathcal{O}'_{\Delta',\ell'}}$

# Setting the Stage: 5-point Conformal Blocks (cont.)

This gives

$$\langle \phi_{\Delta_1}(X_1)\phi_{\Delta_2}(X_2)|\mathcal{O}_{\Delta,\ell}|\Phi_{\Delta_3}(X_3)|\mathcal{O}'_{\Delta',\ell'}|\phi_{\Delta_4}(X_4)\phi_{\Delta_5}(X_5)\rangle = \sum_a \lambda_{\phi_{\Delta_1}\phi_{\Delta_2}\mathcal{O}_{\Delta,\ell}} \lambda^a_{\mathcal{O}_{\Delta,\ell}\Phi_{\Delta_3}\mathcal{O}'_{\Delta',\ell'}} \lambda_{\phi_{\Delta_4}\phi_{\Delta_5}\mathcal{O}'_{\Delta',\ell'}} W^{(a)}_{\Delta,\ell,\Delta',\ell';\Delta_i}(X_i)\,,$$

where

$$W^{(a)}_{\Delta,\ell,\Delta',\ell';\Delta_i}(X_i) = P_{\Delta_i}(X_i)G^{(a)}_{\Delta,\ell,\Delta',\ell'}(u_i)$$

The object $W^{(a)}_{\Delta,\ell,\Delta',\ell';\Delta_i}(X_i)$ is comprised of

- external-dimension-dependent prefactor $P_{\Delta_i}(X_i)$
- 5-point conformal block for arbitrary symmetric traceless exchange $[\Delta,\ell]$, $[\Delta',\ell']$: $G^{(a)}_{\Delta,\ell,\Delta',\ell'}(u_i)$

# Setting the Stage: 5-point Conformal Blocks (cont.)

In 5-point case,

- There are generically five independent conformal cross-ratios $u_i$ for $d \geq 3$
- Can make different choices of basis for $u_i$
- Multiple forms for $P_{\Delta_i}(X_i)$ exist

# Our Conventions

Various conventions for the leg factor and cross-ratios exist in the literature, e.g. in Parikh (2019)

$$P_{\Delta_i}(X_i) = \left(\frac{X_{25}}{X_{15}X_{12}}\right)^{\frac{\Delta_1}{2}} \left(\frac{X_{14}}{X_{15}X_{45}}\right)^{\frac{\Delta_5}{2}} \left(\frac{X_{15}}{X_{12}X_{25}}\right)^{\frac{\Delta_2}{2}} \left(\frac{X_{15}}{X_{13}X_{35}}\right)^{\frac{\Delta_3}{2}} \left(\frac{X_{15}}{X_{14}X_{45}}\right)^{\frac{\Delta_4}{2}},$$

where

$$u_1 = \frac{X_{12}X_{35}}{X_{25}X_{13}}, \quad u_2 = \frac{X_{13}X_{45}}{X_{35}X_{14}}, \quad w_{2;3} = \frac{X_{15}X_{23}}{X_{25}X_{13}}, \quad w_{2;4} = \frac{X_{15}X_{24}}{X_{25}X_{14}}, \quad w_{3;4} = \frac{X_{15}X_{34}}{X_{35}X_{14}}$$

$\Rightarrow$ Here we choose to work in a convention-independent way as much as possible.

# How to Compute the Blocks?

Some prominent methods for computing conformal blocks are

- Conformal integral approach (e.g. Dolan & Osborn (2001, 2004), Simmons-Duffin (2012))
- Conformal Casimir equation (e.g. Dolan & Osborn (2004, 2011), Isachenkov & V. Schomerus (2016), Kravchuk (2018))
- Weight-Shifting operator formalism (e.g. Karateev et al. (2017), Costa & Hansen (2018), Kravchuk & Simmons-Duffin (2018), Karateev et. al. (2018), Albayrak et. al. (2020))

We choose the weight-shifting formalism, which

- Empowers us to derive a set of recursion relations for generating $G^{(a)}_{\Delta,\ell,\Delta',\ell'}(u_i)$

# The Weight-Shifting Operator Formalism

This formalism (due to Karateev et al. (2017) ) introduces a

- Large class of conformally-covariant differential operators

$\Rightarrow$ These operators may be used to relate correlation functions of operators in different representations of the conformal group

$\Rightarrow$ Method enables determination of seed conformal blocks as well as more general blocks

$\Rightarrow$ Allows for efficient derivation of recursion relations

# The Weight-Shifting Operators

Weight-Shifting operators

$\Rightarrow$ Correspond to tensor products of different finite-dimensional representations $\mathcal{W}$

- Each set $\{\mathcal{D}_x^{(v)A}\}$ associated with a particular $\mathcal{W}$
- $A = 1, \ldots, \dim \mathcal{W}$ is an index for $\mathcal{W}$
- $v$ refers to a weight vector of $\mathcal{W}$
- E.g. $\mathcal{W}$ may be the fundamental vector representation $\mathcal{W} = \mathcal{V} = \square$

# The Weight-Shifting Operators (cont.)

In particular,

- $\mathcal{D}_x^{(v)A} : [\Delta, \rho] \to [\Delta - \delta\Delta_v, \lambda]$ associated with $\mathcal{W}$ for generic $\Delta$ are in one-to-one correspondence with irreducible components of $\mathcal{W}^* \otimes V_{\Delta,\rho}$

where $V_{\Delta,\rho}$ is the representation under which $\mathcal{O}(x)$ transforms

$\Rightarrow$ Action of $\mathcal{D}_x^{(v)A}$ on $\mathcal{O}(x)$: to shift the weights of $\mathcal{O}$ by the weights of $v$, while introducing a free $A$ index

For example, to increase or decrease the spin or dimension of $\mathcal{O}$

# The Weight-Shifting Operators (cont.)

- May construct such operators explicitly in the embedding space formalism
- Focus on case of symmetric traceless tensors of $SO(d)$

For vector representation $\mathcal{W} = \mathcal{V}$, can build $\{\mathcal{D}_X^{(\delta\Delta,\delta\ell)A}\}$ which map

$$\begin{aligned}
\mathcal{D}_X^{(-0)A} &: [\Delta, \ell] \to [\Delta - 1, \ell]\,, \\
\mathcal{D}_X^{(0+)A} &: [\Delta, \ell] \to [\Delta, \ell + 1]\,, \\
\mathcal{D}_X^{(0-)A} &: [\Delta, \ell] \to [\Delta, \ell - 1]\,, \\
\mathcal{D}_X^{(+0)A} &: [\Delta, \ell] \to [\Delta + 1, \ell]\,.
\end{aligned}$$

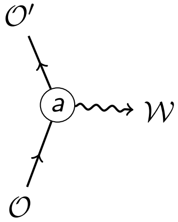
# Crossing Relations for Weight-Shifting Operators

A crucial aspect is that

- Such operators obey a type of crossing relation
- Comes in two varieties: two- and three-point
- Role: to relate action of weight-shifting operators at different points

Symbolize a weight-shifting differential operator by

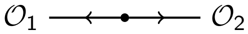
$$\mathcal{D}_X^{(a)A} \quad = \tag{1}$$



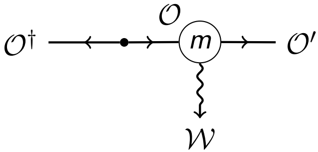
# 2-point Crossing Relation

Represent a conformally-invariant 2-point structure by

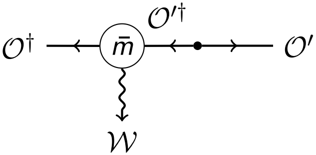
$$\langle \mathcal{O}_1(X_1)\mathcal{O}_2(X_2)\rangle \quad = $$




Acting with a weight-shifting operator on $\langle \mathcal{O}_1(X_1)\mathcal{O}_2(X_2)\rangle$ gives a crossing relation




$$= \begin{Bmatrix} \mathcal{O}^\dagger \\ \mathcal{O}' \end{Bmatrix}^{(m)}_{(\bar{m})}$$




which corresponds to

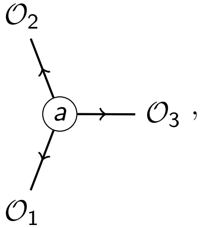
$$\mathcal{D}^{(m)A}_{X_2}\langle \mathcal{O}(X_1)\mathcal{O}(X_2)\rangle = \begin{Bmatrix} \mathcal{O}^\dagger \\ \mathcal{O}' \end{Bmatrix}^{(m)}_{(\bar{m})} \mathcal{D}^{(\bar{m})A}_{X_1}\langle \mathcal{O}'(X_1)\mathcal{O}'(X_2)\rangle$$

where $\bar{m}$ denotes shift opposite to $m$

# 3-point Crossing Relation

Represent a conformally invariant 3-point structure by the vertex

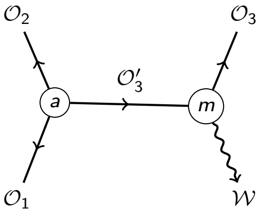
$$\langle \mathcal{O}_1(X_1)\mathcal{O}_2(X_2)\mathcal{O}_3(X_3)\rangle^{(a)} \quad = $$



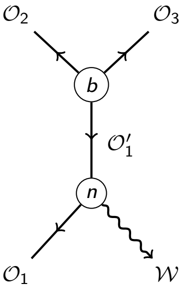

where $a$ enumerates all singlets in $(\rho_1 \otimes \rho_2 \otimes \rho_3)^{SO(d-1)}$

# 3-point Crossing Relation (cont.)

Again, acting on $\langle \mathcal{O}_1(X_1)\mathcal{O}_2(X_2)\mathcal{O}_3(X_3)\rangle^{(a)}$ with a weight-shifting operator gives a crossing relation




$$= \sum_{\mathcal{O}_1', b, n} \begin{Bmatrix} \mathcal{O}_1 & \mathcal{O}_2 & \mathcal{O}_1' \\ \mathcal{O}_3 & \mathcal{W} & \mathcal{O}_3' \end{Bmatrix}^{(a)(m)}_{(b)(n)}$$




which corresponds to

$$\mathcal{D}_{X_3}^{(m)A}\langle \mathcal{O}_1(X_1)\mathcal{O}_2(X_2)\mathcal{O}_3'(X_3)\rangle^{(a)}$$

$$= \sum_{\mathcal{O}_1', b, n} \begin{Bmatrix} \mathcal{O}_1 & \mathcal{O}_2 & \mathcal{O}_1' \\ \mathcal{O}_3 & \mathcal{W} & \mathcal{O}_3' \end{Bmatrix}^{(a)(m)}_{(b)(n)} \mathcal{D}_{X_1}^{(n)A}\langle \mathcal{O}_1'(X_1)\mathcal{O}_2(X_2)\mathcal{O}_3(X_3)\rangle^{(b)}$$

$\Rightarrow$ Coefficients – Racah coefficients or $6j$ symbols

# 3-point Crossing Relation (cont.)

Three-point crossing relation is

⇒ Effectively a change-of-basis equation between different bases of covariant 3-point structures

- Bases generated by the action of a weight-shifting operator at a given point $X_1$ or $X_3$
- Sum over $\mathcal{O}_1'$ is finite, ranging over the operators in $\mathcal{O}_1 \otimes \mathcal{W}$
- Relation reduces to 2-point variety if $\mathcal{O}_2 = \mathbb{1}$

⇒ Relation empowers us to move weight-shifting operators from one leg (operator) to another

⇒ Main computational tool in the formalism!

# Bubble Coefficients

If we contract both sides of the 3-point relation $\mathcal{D}^{(n)}_{X_1\,A}$, find

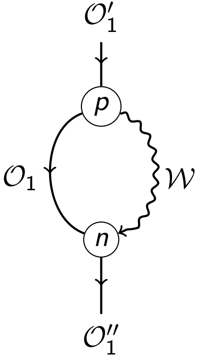
$$\mathcal{D}^{(n)}_{X_1\,A}\mathcal{D}^{(m)A}_{X_3}\langle\mathcal{O}_1(X_1)\mathcal{O}_2(X_2)\mathcal{O}_3'(X_3)\rangle^{(a)}$$
$$=\sum_{\mathcal{O}_1',b,p}\begin{Bmatrix}\mathcal{O}_1 & \mathcal{O}_2 & \mathcal{O}_1'\\ \mathcal{O}_3 & \mathcal{W} & \mathcal{O}_3'\end{Bmatrix}^{(a)(m)}_{(b)(p)}\mathcal{D}^{(n)}_{X_1\,A}\mathcal{D}^{(p)A}_{X_1}\langle\mathcal{O}_1'(X_1)\mathcal{O}_2(X_2)\mathcal{O}_3(X_3)\rangle^{(b)}\,.$$

$\Rightarrow$ RHS features two contracted weight-shifting operators acting at the same point!

# Bubble Coefficients (cont.)

- Composition $\mathcal{D}^{(n)}_{X_1\,A}\mathcal{D}^{(p)A}_{X_1}$ corresponds to a bubble diagram:

$$\mathcal{D}^{(n)}_{X_1\,A}\mathcal{D}^{(p)A}_{X_1} =$$




$$= \begin{pmatrix} \mathcal{O}'_1 \\ \mathcal{O}_1 \;\; \mathcal{W} \end{pmatrix}^{(n)(p)} \delta_{\mathcal{O}'_1\mathcal{O}''_1}$$

# Gluing 3-point Functions to Form Conformal Blocks

Standard way to encode a conformal block:

- Conformal integral of product of 3-point functions

E.g. scalar exchange block in a purely scalar 4-point function has the form

$$\frac{1}{\mathcal{N}_{\mathcal{O}}}\int D^d X D^d Y \langle \phi_{\Delta_1}(X_1)\phi_{\Delta_2}(X_2)\mathcal{O}(X)\rangle \frac{1}{(-2X\cdot Y)^{d-\Delta}}\langle \mathcal{O}(Y)\phi_{\Delta_3}(X_3)\phi_{\Delta_4}(X_4)\rangle\Big|_M$$

with $M = e^{2\pi i\varphi}$ denoting the projection onto the appropriate monodromy invariant subspace

# Gluing 3-point Functions to Form Conformal Blocks (cont.)

In the weight-shifting formalism (Karateev et al. (2017) ),

- Operation which "glues" the 3-point correlators $\langle \phi_{\Delta_1}(X_1)\phi_{\Delta_2}(X_2)\mathcal{O}(X)\rangle$ and $\langle \mathcal{O}(Y)\phi_{\Delta_3}(X_3)\phi_{\Delta_4}(X_4)\rangle$ together

Symbolized by

$$|\mathcal{O}\rangle \bowtie \langle \mathcal{O}| \equiv \frac{1}{\mathcal{N}_{\mathcal{O}}} \int D^d X D^d Y |\mathcal{O}(X)\rangle \frac{1}{(-2X \cdot Y)^{d-\Delta}} \langle \mathcal{O}(Y)|$$

$$= \mathcal{O} \dashrightarrow\!\!\succ \dashrightarrow \times \dashleftarrow \prec\!\!\dashleftarrow \mathcal{O} \cdot$$

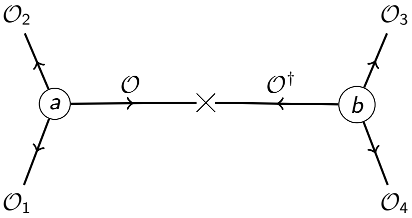
For spinning operators,

- $\mathcal{O}_{\Delta,\rho}$ to be glued to representation with which it has a nonvanishing 2-point function

# Gluing 3-point Functions to Form Conformal Blocks (cont.)

In terms of this notation, a general 4-point conformal block is given by

$$W^{ab} \equiv \langle \mathcal{O}_1 \mathcal{O}_2 \mathcal{O} \rangle^{(a)} \bowtie {}^{(b)}\langle \mathcal{O}^\dagger \mathcal{O}_3 \mathcal{O}_4 \rangle =$$



# General Strategy

Our overall strategy involves

⇒ Acting with specific combinations of weight-shifting operators on a given conformal block

⇒ Then applying the two- and three- point crossing relations as needed

Goal: to re-express the original block in terms of

- linear combinations of lower-spin blocks with shifted external and, potentially, exchanged dimensions

# General Strategy (cont.)

- To implement such forms, require a mechanism for integrating by parts

This is the statement

$$|\mathcal{D}^{(c)A}\mathcal{O}\rangle \bowtie \langle\mathcal{O}'^{\dagger}| = \sum_m \begin{Bmatrix} \mathcal{O}^{\dagger} & \mathbb{1} & \mathcal{O}'^{\dagger} \\ \mathcal{O}' & \mathcal{W} & \mathcal{O} \end{Bmatrix}^{\bullet(c)}_{\bullet(m)} |\mathcal{O}\rangle \bowtie \langle\mathcal{D}^{(m)A}\mathcal{O}'^{\dagger}|$$

- Empowers us to move the weight-shifting operators from one side of the $\bowtie$ to the other!

# Recursion Relations from Weight-Shifting Operators: Four-Point Case

Describe the basic procedure for extracting recursion relations:

- Four-point scalar conformal blocks defined as

$$\langle \phi_{\Delta_1}(X_1)\phi_{\Delta_2}(X_2)|\mathcal{O}_{\Delta,\ell}|\phi_{\Delta_3}(X_3)\phi_{\Delta_4}(X_4)\rangle = \frac{1}{(X_{12})^{\frac{1}{2}(\Delta_1+\Delta_2)}(X_{34})^{\frac{1}{2}(\Delta_3+\Delta_4)}} \times \left(\frac{X_{24}}{X_{14}}\right)^{\Delta_{12}/2}\left(\frac{X_{14}}{X_{13}}\right)^{\Delta_{34}/2} G_{\Delta,\ell}(u,v),$$

where $\Delta_{ij} = \Delta_i - \Delta_j$

- Act on this object with the combination of operators

$$-2(\mathcal{D}_{X_1}^{(-0)} \cdot \mathcal{D}_{X_4}^{(-0)}) = -2X_1 \cdot X_4 = X_{14}$$

# Recursion Relations from Weight-Shifting Operators: Four-Point Case (cont.)

- Gives a 4-point function with $\Delta_1 \to \Delta_1 - 1$ and $\Delta_4 \to \Delta_4 - 1$
- Shifts in $\Delta_1$ and $\Delta_4$ $\Rightarrow$ a shifted external prefactor
- Absorb it into $u^{-1/2}$

Next, apply

$\Rightarrow$ three-point crossing relation

$\Rightarrow$ integration-by-parts rule

In three-point rule, sum over

$$\square \otimes [\Delta, \ell] = [\Delta - 1, \ell] \oplus [\Delta, \ell + 1] \oplus [\Delta, \ell - 1] \oplus [\Delta + 1, \ell] + \ldots .$$

# Recursion Relations from Weight-Shifting Operators: Four-Point Case (cont.)

Result is the familiar recursion relation due to Dolan and Osborn:

$$G_{\Delta,\ell}(u,v) = \frac{1}{s^{(14)}}\left( u^{-1/2} G_{\Delta,\ell-1}(u,v)\Big|_{\Delta_1\to\Delta_1+1,\Delta_4\to\Delta_4+1} - G_{\Delta-1,\ell-1}(u,v) - t^{(14)} G_{\Delta,\ell-2}(u,v) - u^{(14)} G_{\Delta+1,\ell-1}(u,v)\right)$$

$\Rightarrow$ This is Eq. (4.18) in Dolan & Osborn (2011)

Now wish to generalize this analysis to 5-point functions!

# Mapping out the Derivation of the 5-point Recursion Relations

Basic idea: to express 5-point conformal block for $([\Delta, \ell], [\Delta', \ell'])$ exchange in terms of lower-spin blocks

- As before, act on 5-point function

$$\begin{aligned}&\langle \phi_{\Delta_1}(X_1)\phi_{\Delta_2}(X_2)|\mathcal{O}_{\Delta,\ell}|\Phi_{\Delta_3}(X_3)|\mathcal{O}'_{\Delta',\ell'}|\phi_{\Delta_4}(X_4)\phi_{\Delta_5}(X_5)\rangle \\ &\qquad = \langle \phi_{\Delta_1}(X_1)\phi_{\Delta_2}(X_2)\mathcal{O}_{\Delta,\ell}\rangle \bowtie \langle \mathcal{O}_{\Delta,\ell}\Phi_{\Delta_3}(X_3)\mathcal{O}'_{\Delta',\ell'}\rangle \bowtie \langle \mathcal{O}'_{\Delta',\ell'}\phi_{\Delta_4}(X_4)\phi_{\Delta_5}(X_5)\rangle \\ &\qquad = \sum_a \sum_{\mathcal{O}_{\Delta,\ell}} \sum_{\mathcal{O}'_{\Delta',\ell'}} \lambda_{\phi_{\Delta_1}\phi_{\Delta_2}\mathcal{O}_{\Delta,\ell}} \lambda^a_{\mathcal{O}_{\Delta,\ell}\Phi_{\Delta_3}\mathcal{O}'_{\Delta',\ell'}} \lambda_{\phi_{\Delta_4}\phi_{\Delta_5}\mathcal{O}'_{\Delta',\ell'}} W^{(a)}_{\Delta,\ell,\Delta',\ell';\Delta_i}(X_i)\end{aligned}$$

With weight-shifting operator combination

$$\begin{aligned}&-2(\mathcal{D}^{(-0)}_{X_1} \cdot \mathcal{D}^{(-0)}_{X_3})\langle \phi_{\Delta_1}(X_1)\phi_{\Delta_2}(X_2)|\mathcal{O}_{\Delta,\ell}|\Phi_{\Delta_3}(X_3)|\mathcal{O}'_{\Delta',\ell'}|\phi_{\Delta_4}(X_4)\phi_{\Delta_5}(X_5)\rangle \\ &\qquad = X_{13}\langle \phi_{\Delta_1}(X_1)\phi_{\Delta_2}(X_2)|\mathcal{O}_{\Delta,\ell}|\Phi_{\Delta_3}(X_3)|\mathcal{O}'_{\Delta',\ell'}|\phi_{\Delta_4}(X_4)\phi_{\Delta_5}(X_5)\rangle\end{aligned}$$

# Mapping out the Derivation of the 5-point Recursion Relations (cont.)

Consider

$$\langle \phi_{\Delta_1}(X_1)\phi_{\Delta_2}(X_2)\mathcal{O}_{\Delta,\ell}\rangle \bowtie \langle \mathcal{O}_{\Delta,\ell}\Phi_{\Delta_3}(X_3)\mathcal{O}'_{\Delta',\ell'}\rangle \bowtie \langle \mathcal{O}'_{\Delta',\ell'}\phi_{\Delta_4}(X_4)\phi_{\Delta_5}(X_5)\rangle$$

Apply three-point crossing relation to $\langle \phi_{\Delta_1}(X_1)\phi_{\Delta_2}(X_2)\mathcal{O}_{\Delta,\ell}\rangle$:

$$\begin{aligned}\mathcal{D}_{X_1}^{(-0)A}\langle \phi_{\Delta_1}(X_1)\phi_{\Delta_2}(X_2)\mathcal{O}_{\Delta,\ell}(X_I)\rangle = {} & \mathcal{A}_{(+0)}^{(-0)}\mathcal{D}_{X_I}^{(+0)A}\langle \phi_{\Delta_1-1}(X_1)\phi_{\Delta_2}(X_2)\mathcal{O}_{\Delta-1,\ell}(X_I)\rangle \\ & +\mathcal{A}_{(0-)}^{(-0)}\mathcal{D}_{X_I}^{(0-)A}\langle \phi_{\Delta_1-1}(X_1)\phi_{\Delta_2}(X_2)\mathcal{O}_{\Delta,\ell+1}(X_I)\rangle \\ & +\mathcal{A}_{(0+)}^{(-0)}\mathcal{D}_{X_I}^{(0+)A}\langle \phi_{\Delta_1-1}(X_1)\phi_{\Delta_2}(X_2)\mathcal{O}_{\Delta,\ell-1}(X_I)\rangle \\ & +\mathcal{A}_{(-0)}^{(-0)}\mathcal{D}_{X_I}^{(-0)A}\langle \phi_{\Delta_1-1}(X_1)\phi_{\Delta_2}(X_2)\mathcal{O}_{\Delta+1,\ell}(X_I)\rangle\end{aligned}$$

May extract $6j$ symbols $\mathcal{A}_{(n)}^{(-0)}$ by

- Acting on both sides with $\mathcal{D}_{X_I\,A}^{(\bar{n})}$ ($\bar{n}$ has shift opposite to $n$)
- Noting $\exists$ only one nonzero bubble coefficient on RHS
- Isolating $\mathcal{A}_{(n)}^{(-0)}$

# Mapping out the Derivation of the 5-point Recursion Relations (cont.)

Next step is

- To push each of the operators $\mathcal{D}_{X_I}^{(n)A}$ through the shadow integral

For this, invoke integration-by-parts rule to move $\mathcal{D}_{X_I}^{(n)A}$ across $\bowtie$!

- For example, for $\mathcal{D}_{X_I}^{(+0)A}$

$$|\mathcal{D}_{X_I}^{(+0)A}\mathcal{O}_{\Delta-1,\ell}\rangle \bowtie \langle\mathcal{O}_{\Delta,\ell}| = B_{(+0)(-0)}|\mathcal{O}_{\Delta-1,\ell}\rangle \bowtie \langle\mathcal{D}_{X_I}^{(-0)A}\mathcal{O}_{\Delta,\ell}|$$

# Mapping out the Derivation of the 5-point Recursion Relations (cont.)

At this point, arrive at

$$
\begin{aligned}
&\mathcal{D}_{X_1}^{(-0)A}\langle\phi_{\Delta_1}(X_1)\phi_{\Delta_2}(X_2)\mathcal{O}_{\Delta,\ell}(X_I)\rangle \bowtie \langle\mathcal{O}_{\Delta,\ell}(X_I)\Phi_{\Delta_3}(X_3)\mathcal{O}'_{\Delta',\ell'}(X_J)\rangle^{(a)} = \\
&\mathcal{A}_{(+0)}^{(-0)}B_{(+0)(-0)}\langle\phi_{\Delta_1-1}(X_1)\phi_{\Delta_2}(X_2)\mathcal{O}_{\Delta-1,\ell}(X_I)\rangle \bowtie \mathcal{D}_{X_I}^{(-0)A}\langle\mathcal{O}_{\Delta,\ell}(X_I)\Phi_{\Delta_3}(X_3)\mathcal{O}'_{\Delta',\ell'}(X_J)\rangle^{(a)} \\
&+\mathcal{A}_{(0-)}^{(-0)}B_{(0-)(0+)}\langle\phi_{\Delta_1-1}(X_1)\phi_{\Delta_2}(X_2)\mathcal{O}_{\Delta,\ell+1}(X_I)\rangle \bowtie \mathcal{D}_{X_I}^{(0+)A}\langle\mathcal{O}_{\Delta,\ell}(X_I)\Phi_{\Delta_3}(X_3)\mathcal{O}'_{\Delta',\ell'}(X_J)\rangle^{(a)} \\
&+\mathcal{A}_{(0+)}^{(-0)}B_{(0+)(0-)}\langle\phi_{\Delta_1-1}(X_1)\phi_{\Delta_2}(X_2)\mathcal{O}_{\Delta,\ell-1}(X_I)\rangle \bowtie \mathcal{D}_{X_I}^{(0-)A}\langle\mathcal{O}_{\Delta,\ell}(X_I)\Phi_{\Delta_3}(X_3)\mathcal{O}'_{\Delta',\ell'}(X_J)\rangle^{(a)} \\
&+\mathcal{A}_{(-0)}^{(-0)}B_{(-0)(+0)}\langle\phi_{\Delta_1-1}(X_1)\phi_{\Delta_2}(X_2)\mathcal{O}_{\Delta+1,\ell}(X_I)\rangle \bowtie \mathcal{D}_{X_I}^{(+0)A}\langle\mathcal{O}_{\Delta,\ell}(X_I)\Phi_{\Delta_3}(X_3)\mathcal{O}'_{\Delta',\ell'}(X_J)\rangle^{(a)}
\end{aligned}
$$

Next apply the three-point relation again!

⇒ Purpose: to move the action of $\mathcal{D}_{X_I}^{(n)A}$ from the internal point $X_I$ to the external point $X_3$!

- Shifts $[\Delta_3-\delta\Delta_n,-\delta\ell_n]$ take on values in

$$\square\otimes[\Delta_3,0]=[\Delta_3-1,0]\oplus[\Delta_3,1]\oplus[\Delta_3+1,0]$$

# Mapping out the Derivation of the 5-point Recursion Relations (cont.)

At this stage, recall that

- $\mathcal{D}_{X_1}^{(-0)A}$ is contracted with $\mathcal{D}_{X_3\,A}^{(-0)}$ in our combination of choice

$\Rightarrow$ So all bubble coefficients on RHS vanish except for one, $\mathcal{D}_{X_3\,A}^{(-0)}\mathcal{D}_{X_3}^{(+0)A}$

- Label $a$ enumerates constituent 3-point tensor structures of $\langle\mathcal{O}_{\Delta,\ell}\Phi_{\Delta_3}\mathcal{O}'_{\Delta',\ell'}\rangle$, i.e. $\langle\mathcal{O}_{\Delta,\ell}\Phi_{\Delta_3}\mathcal{O}'_{\Delta',\ell'}\rangle^{(a)}$

- Parameterize structures by the index $n_{IJ}$: $0\leq n_{IJ}\leq\min(\ell,\ell')$

# Mapping out the Derivation of the 5-point Recursion Relations (cont.)

Extracting relevant $6j$ symbols and combining everything, obtain

$$
\begin{aligned}
-2(\mathcal{D}_{X_1}^{(-0)} \cdot \mathcal{D}_{X_3}^{(-0)}) W^{(n_{IJ})}_{\Delta,\ell,\Delta',\ell';\Delta_1,\Delta_2,\Delta_3,\Delta_4,\Delta_5} = \\
& - 2b_{\Phi}^{(-0)(+0)} \Bigg( \mathcal{A}^{(-0)}_{(+0)} B_{(+0)(-0)} \mathcal{B}^{n_{IJ}(-0)}_{n_{IJ}(+0)} W^{(n_{IJ})}_{\Delta-1,\ell,\Delta',\ell';\Delta_1-1,\Delta_2,\Delta_3-1,\Delta_4,\Delta_5} \\
& + \sum_{m_{IJ}=n_{IJ}}^{n_{IJ}+1} \mathcal{A}^{(-0)}_{(0-)} B_{(0-)(0+)} \mathcal{B}^{n_{IJ}(0+)}_{m_{IJ}(+0)} W^{(m_{IJ})}_{\Delta,\ell+1,\Delta',\ell';\Delta_1-1,\Delta_2,\Delta_3-1,\Delta_4,\Delta_5} \\
& + \sum_{m_{IJ}=n_{IJ}-1}^{n_{IJ}+1} \mathcal{A}^{(-0)}_{(0+)} B_{(0+)(0-)} \mathcal{B}^{n_{IJ}(0-)}_{m_{IJ}(+0)} W^{(m_{IJ})}_{\Delta,\ell-1,\Delta',\ell';\Delta_1-1,\Delta_2,\Delta_3-1,\Delta_4,\Delta_5} \\
& + \sum_{m_{IJ}=n_{IJ}-1}^{n_{IJ}+2} \mathcal{A}^{(-0)}_{(-0)} B_{(-0)(+0)} \mathcal{B}^{n_{IJ}(+0)}_{m_{IJ}(+0)} W^{(m_{IJ})}_{\Delta+1,\ell,\Delta',\ell';\Delta_1-1,\Delta_2,\Delta_3-1,\Delta_4,\Delta_5} \Bigg)
\end{aligned}
$$

# Mapping out the Derivation of the 5-point Recursion Relations (cont.)

$\Rightarrow$ Evidently a recursion relation in spin $\ell$, with $\ell'$ held fixed!

Next apply analogous approach to the other spin:

$$-2(\mathcal{D}_{X_3}^{(-0)} \cdot \mathcal{D}_{X_5}^{(-0)})\langle\phi_{\Delta_1}(X_1)\phi_{\Delta_2}(X_2)|\mathcal{O}_{\Delta,\ell}|\Phi_{\Delta_3}(X_3)|\mathcal{O}'_{\Delta',\ell'}|\phi_{\Delta_4}(X_4)\phi_{\Delta_5}(X_5)\rangle$$
$$= X_{35}\langle\phi_{\Delta_1}(X_1)\phi_{\Delta_2}(X_2)|\mathcal{O}_{\Delta,\ell}|\Phi_{\Delta_3}(X_3)|\mathcal{O}'_{\Delta',\ell'}|\phi_{\Delta_4}(X_4)\phi_{\Delta_5}(X_5)\rangle$$

Mirror image of the above procedure with

- $\Delta \leftrightarrow \Delta'$
- $\ell \leftrightarrow \ell'$
- $\Delta_{12} \rightarrow -\Delta_{45}$

# Main Results

For convenience, adopt shorthand notation

$$G^{(n)}_{(\ell,\ell';\delta_0,\delta_0')} \equiv G^{(n)}_{\Delta+\delta_0,\ell,\Delta'+\delta_0',\ell'}(u_i)$$

Shifting $\Delta_3 \to \Delta_3 + 1$, and $\Delta_1 \to \Delta_1 + 1$, $\ell \to \ell - 1$; $\Delta_5 \to \Delta_5 + 1$, $\ell' \to \ell' - 1$, obtain the set

$$(1)\ G^{(n_{IJ})}_{(\ell,\ell';0,0)} = \frac{1}{s_{n_{IJ}}}\Big( f(u_i) G^{(n_{IJ})}_{(\ell-1,\ell';0,0)}\Big|_{\Delta_1\to\Delta_1+1,\Delta_3\to\Delta_3+1} - G^{(n_{IJ})}_{(\ell-1,\ell';-1,0)} - s_{n_{IJ}+1} G^{(n_{IJ}+1)}_{(\ell,\ell';0,0)}$$
$$-t_{n_{IJ}-1} G^{(n_{IJ}-1)}_{(\ell-2,\ell';0,0)} - t_{n_{IJ}} G^{(n_{IJ})}_{(\ell-2,\ell';0,0)} - t_{n_{IJ}+1} G^{(n_{IJ}+1)}_{(\ell-2,\ell';0,0)}$$
$$-u_{n_{IJ}-1} G^{(n_{IJ}-1)}_{(\ell-1,\ell';1,0)} - u_{n_{IJ}} G^{(n_{IJ})}_{(\ell-1,\ell';1,0)} - u_{n_{IJ}+1} G^{(n_{IJ}+1)}_{(\ell-1,\ell';1,0)} - u_{n_{IJ}+2} G^{(n_{IJ}+2)}_{(\ell-1,\ell';1,0)}\Big)$$

and

$$(2)\ G^{(n_{IJ})}_{(\ell,\ell';0,0)} = \frac{1}{s'_{n_{IJ}}}\Big( f'(u_i) G^{(n_{IJ})}_{(\ell,\ell'-1;0,0)}\Big|_{\Delta_3\to\Delta_3+1,\Delta_5\to\Delta_5+1} - G^{(n_{IJ})}_{(\ell,\ell'-1;0,-1)} - s'_{n_{IJ}+1} G^{(n_{IJ}+1)}_{(\ell,\ell';0,0)}$$
$$-t'_{n_{IJ}-1} G^{(n_{IJ}-1)}_{(\ell,\ell'-2;0,0)} - t'_{n_{IJ}} G^{(n_{IJ})}_{(\ell,\ell'-2;0,0)} - t'_{n_{IJ}+1} G^{(n_{IJ}+1)}_{(\ell,\ell'-2;0,0)}$$
$$-u'_{n_{IJ}-1} G^{(n_{IJ}-1)}_{(\ell,\ell'-1;0,1)} - u'_{n_{IJ}} G^{(n_{IJ})}_{(\ell,\ell'-1;0,1)} - u'_{n_{IJ}+1} G^{(n_{IJ}+1)}_{(\ell,\ell'-1;0,1)} - u'_{n_{IJ}+2} G^{(n_{IJ}+2)}_{(\ell,\ell'-1;0,1)}\Big)$$

# Discussion of Results

Relations defined in a convention-independent way:

- $f(u_i)$ and $f'(u_i)$ represent cross-ratio-dependent prefactors
- e.g. for the set of conventions in Parikh (2019)

$$f(u_i) = (u_1)^{-1/2}, \quad f'(u_i) = (u_2)^{-1/2}$$

- Coefficients − products of the various $6j$ symbols

May regard above relations as two independent results:

- One re-expresses a block for $[\Delta, \ell]$, $[\Delta', \ell']$ exchange in terms of $\{(\ell, \ell'), (\ell - 1, \ell'), (\ell - 2, \ell')\}$
- The other does the same for $\ell'$, with $\ell$ held fixed

# Discussion of Results (cont.)

**Remark**: All terms on RHS have **lower spins** except

- $s_{n_{IJ}+1}$ and $s'_{n_{IJ}+1}$

But these have a **larger** 3-point structure index, $n_{IJ}+1$ and vanish **only**

- $s_{n_{IJ}+1}$: at maximum value $n_{IJ} = \min(\ell - 1, \ell') = \ell'$ for $\ell' < \ell$
- $s'_{n_{IJ}+1}$: at maximum value $n_{IJ} = \min(\ell, \ell' - 1) = \ell$ for $\ell < \ell'$

**Observe:** Case $\ell = \ell'$ and $n_{IJ} = \ell - 1$ missing here

- Need additional relation for this!

# Discussion of Results (cont.)

Hence, generate the blocks as follows:

- If $\ell' \leq \ell$, start from the seed $n_{IJ} = \ell'$ and iterate (1) until $\ell' > \ell$
- If $\ell \leq \ell'$, start from the seed $n_{IJ} = \ell$ and iterate (2) until $\ell > \ell'$
- If $\ell' = \ell$ and $n_{IJ} = \ell - 1$, use special recursion relation (combine with action of $-2(\mathcal{D}_{X_1}^{(-0)} \cdot \mathcal{D}_{X_5}^{(-0)})$)

Thus, can use these relations together

$\Rightarrow$ to recursively generate 5-point conformal blocks for arbitrary $[\ell, \ell']$ exchange, starting from the seeds $\ell = \ell' = 0$

To sum up:

- Given an explicit prescription for arbitrary $[\ell, \ell']$ exchange 5-point blocks

# Promoting $\Phi_{\Delta_3}$ to a Spinning Operator: Spin 1

Seek to promote the middle operator $\Phi_{\Delta_3}$ to a spinning operator

- Simplest case: $\Phi_{\Delta_3}$ to vector operator

Goal:

- To cast $(\mathcal{O}_{\Delta,\ell}, \mathcal{O}'_{\Delta',\ell'})$ exchange block in terms of seed blocks for purely scalar 5-point function

In

$$\langle \phi_{\Delta_1}(X_1)\phi_{\Delta_2}(X_2)\mathcal{O}_{\Delta,\ell}\rangle \bowtie \langle \mathcal{O}_{\Delta,\ell}\Phi_{\Delta_3}(X_3)\mathcal{O}'_{\Delta',\ell'}\rangle \bowtie \langle \mathcal{O}'_{\Delta',\ell'}\phi_{\Delta_4}(X_4)\phi_{\Delta_5}(X_5)\rangle$$

Need to take $\langle \mathcal{O}_{\Delta,\ell}\Phi_{\Delta_3}(X_3)\mathcal{O}'_{\Delta',\ell'}\rangle \to \langle \mathcal{O}_{\Delta,\ell} v^A(X_3)\mathcal{O}'_{\Delta',\ell'}\rangle$

# Promoting $\Phi_{\Delta_3}$ to a Spinning Operator: Spin 1 (cont.)

- Have 3 distinct classes of constituent 3-point structures:

$$Q_{(\ell,1,\ell')}(X_1,X_2,X_3;Z_1,Z_2,Z_3)=\sum_{i=1}^{3}\lambda_{i,n_{IJ}}Q_{(\ell,1,\ell')}^{(i,n_{IJ})}\,,$$

where

$$Q_{(\ell,1,\ell')}^{(i,n_{IJ})}=\frac{q_{(\ell,1,\ell')}^{(i,n_{IJ})}}{(X_{12})^{\frac{1}{2}(\Delta+\Delta_3-\Delta'+\ell-\ell'+1)}(X_{13})^{\frac{1}{2}(\Delta-\Delta_3+\Delta'+\ell+\ell'-1)}(X_{23})^{\frac{1}{2}(-\Delta+\Delta_3+\Delta'-\ell+\ell'+1)}}$$

with

$$\begin{aligned}
q_{(\ell,1,\ell')}^{(1,n_{IJ})}&=V_1^{\ell-n_{IJ}}V_2V_3^{\ell'-n_{IJ}}H_{13}^{n_{IJ}}\,,\\
q_{(\ell,1,\ell')}^{(2,n_{IJ})}&=V_1^{\ell-n_{IJ}}V_3^{(\ell'-1)-n_{IJ}}H_{13}^{n_{IJ}}H_{23}\,,\\
q_{(\ell,1,\ell')}^{(3,n_{IJ})}&=V_1^{(\ell-1)-n_{IJ}}V_3^{\ell'-n_{IJ}}H_{12}H_{13}^{n_{IJ}}\,.
\end{aligned}$$

Remark:

- $i=1$ exist for $n_{IJ}\in[0,\min(\ell,\ell')]$
- $i=2$ exist for $n_{IJ}\in[0,\min(\ell,\ell'-1)]$
- $i=3$ exist for $n_{IJ}\in[0,\min(\ell-1,\ell')]$

# Promoting $\Phi_{\Delta_3}$ to a Spinning Operator: Spin 1 (cont.)

Consider the quantity

$$W^{(V)(i,n_{IJ})}_{\Delta,\ell;\Delta',\ell';\Delta_1,\Delta_2,\Delta_3,\Delta_4,\Delta_5} = \langle \phi_{\Delta_1}(X_1)\phi_{\Delta_2}(X_2)\mathcal{O}_{\Delta,\ell}\rangle \bowtie Q^{(i,n_{IJ})}_{(\ell,1,\ell')} \bowtie \langle \mathcal{O}'_{\Delta',\ell'}\phi_{\Delta_4}(X_4)\phi_{\Delta_5}(X_5)\rangle$$

- $i$ enumerates the 3 distinct classes
- $n_{IJ}$ parameterizes different possible structures within each class

Start by expressing $Q^{(i,n_{IJ})}_{(\ell,1,\ell')}$ for fixed $i$ in terms of the basis

$$\{(\mathcal{D}^{(-0)}_{X}\cdot\mathcal{D}^{(0+)}_{X_3}),(\mathcal{D}^{(+0)}_{X}\cdot\mathcal{D}^{(0+)}_{X_3}),(\mathcal{D}^{(0-)}_{X}\cdot\mathcal{D}^{(0+)}_{X_3}),(\mathcal{D}^{(0+)}_{X}\cdot\mathcal{D}^{(0+)}_{X_3})\}$$

for either $X = X_I$ or $X = X_J$

# Promoting $\Phi_{\Delta_3}$ to a Spinning Operator: Spin 1 (cont.)

Here we

- Choose $\mathcal{D}_{X_3}^{(0+)}$ to raise spin of $\Phi_{\Delta_3}$ to 1

For example, for $X = X_I$

$$(\mathcal{D}_{X_I}^{(-0)} \cdot \mathcal{D}_{X_3}^{(0+)})\langle \mathcal{O}_{\Delta+1,\ell}(X_I)\Phi_{\Delta_3}(X_3)\mathcal{O}_{\Delta',\ell'}(X_J)\rangle^{(n_{IJ})}$$
$$= \alpha_1 Q_{(\ell,1,\ell')}^{(1,n_{IJ})} + \beta_1 Q_{(\ell,1,\ell')}^{(2,n_{IJ})} + \gamma_1 Q_{(\ell,1,\ell')}^{(3,n_{IJ})}$$

Now, since only three distinct 3-point structures

$\Rightarrow$ just need three equations:
$\{(\mathcal{D}_{X_I}^{(-0)} \cdot \mathcal{D}_{X_3}^{(0+)}), (\mathcal{D}_{X_I}^{(+0)} \cdot \mathcal{D}_{X_3}^{(0+)}), (\mathcal{D}_{X_I}^{(0-)} \cdot \mathcal{D}_{X_3}^{(0+)})\}$

- reuse these multiple times to generate set involving $Q_{(\ell,1,\ell')}^{(i,n_{IJ}-1)}$, $Q_{(\ell,1,\ell')}^{(i,n_{IJ})}$, and $Q_{(\ell,1,\ell')}^{(i,n_{IJ}+1)}$
- then solve for structures

# Promoting $\Phi_{\Delta_3}$ to a Spinning Operator: Spin 1 (cont.)

We next apply

- three-point crossing relation (a variety that holds $\ell$ fixed)
- integration-by-parts rule

to obtain a set of recursion relations for

- $W^{(V)(i,n_{IJ})}_{\Delta,\ell;\Delta',\ell';\Delta_1,\Delta_2,\Delta_3,\Delta_4,\Delta_5}, \qquad i = 1, 2, 3$

E.g.

$$
\begin{aligned}
W^{(V)(2,n_{IJ})}_{\Delta,\ell;\Delta',\ell';\Delta_1,\Delta_2,\Delta_3,\Delta_4,\Delta_5} &= \frac{(n_{IJ}-\ell)\,(n_{IJ}-\ell'+1)}{(n_{IJ}+1)\,(\Delta'-\Delta+\Delta_3-2n_{IJ}+\ell'+\ell-1)}\,W^{(V)(2,n_{IJ}+1)}_{\Delta,\ell;\Delta',\ell';\Delta_1,\Delta_2,\Delta_3,\Delta_4,\Delta_5}\\
&+\sum_{m=n_{IJ}}^{n_{IJ}+2}\mathscr{B}^{(1)(m)}_{(+0)(0+)}(\mathcal{D}^{(+0)}_{X_1}\cdot\mathcal{D}^{(0+)}_{X_3})W^{(m)}_{\Delta,\ell,\Delta',\ell';\Delta_1-1,\Delta_2,\Delta_3,\Delta_4,\Delta_5}\\
&+\mathscr{B}^{(2)(m)}_{(+0)(0+)}(\mathcal{D}^{(+0)}_{X_2}\cdot\mathcal{D}^{(0+)}_{X_3})W^{(m)}_{\Delta,\ell,\Delta',\ell';\Delta_1,\Delta_2-1,\Delta_3,\Delta_4,\Delta_5}\\
&+\mathscr{B}^{(1)(m)}_{(-0)(0+)}(\mathcal{D}^{(-0)}_{X_1}\cdot\mathcal{D}^{(0+)}_{X_3})W^{(m)}_{\Delta,\ell,\Delta',\ell';\Delta_1+1,\Delta_2,\Delta_3,\Delta_4,\Delta_5}\\
&+\mathscr{B}^{(2)(m)}_{(-0)(0+)}(\mathcal{D}^{(-0)}_{X_2}\cdot\mathcal{D}^{(0+)}_{X_3})W^{(m)}_{\Delta,\ell,\Delta',\ell';\Delta_1,\Delta_2+1,\Delta_3,\Delta_4,\Delta_5}
\end{aligned}
$$

# Comment on Spin 2 Promotion

To promote $\Phi_{\Delta_3}$ to a spin-2 operator $T^{AB}$,

- Follow exactly analogous procedure
- May recycle much of the spin-1 calculation
- Take $W^{(V)(i,n_{IJ})}_{\Delta,\ell;\Delta',\ell';\Delta_1,\Delta_2,\Delta_3,\Delta_4,\Delta_5}$ as the seed blocks

# The Averaged Null Energy Condition (ANEC): An Application

All QFTs known to respect a special positivity condition:

- averaged null energy condition (ANEC)

  Hofman & Maldacena (2008), Faulkner et al. (2016), Hartman et al. (2016)

which states that the energy flux operator

$$\mathcal{E} = \int_{-\infty}^{\infty} dx^- \; T_{--}(x^-, 0) \,,$$

where the integral is over a complete null line, satisfies

$$\langle \Psi | \mathcal{E} | \Psi \rangle \geq 0$$

We ask:

- Can we get novel constraints on OPE coefficients by studying ANEC positivity in five-point functions?

# The Averaged Null Energy Condition (ANEC): An Application (cont.)

Possible application of our results:

- May use the OPE to compute the expectation value of the ANEC operator in bilocal states $\phi(x_1)\phi(x_2)|0\rangle$, encoded in $\langle\phi_i\phi_j T^{\mu\nu}\phi_i\phi_j\rangle$ and demand positivity
- Expect OPE limit $x_{12}, x_{45} \to 0$ to be dominated by stress tensor or low-dimension scalars
- May consider smeared states of the form

$$|\phi_i\phi_j\rangle_f = \int d^d x_1 \int d^d x_2 \; f(x_1, x_2)\phi_i(x_1)\phi_j(x_2)|0\rangle$$

  with $f$ chosen to have support such that convergence of the $\phi_i \times \phi_j$ OPE is preserved
- E.g. $f(x_1, x_2) \propto e^{-iq(t_1+t_2)}$ to correspond to approximate energy eigenstates

Cordova et al. (2017), Meltzer et al. (2018)

# The Averaged Null Energy Condition (ANEC): An Application (cont.)

- May analyze more general mixed states created by linear combinations of operators
- E.g. consider mixing with a state

$$|T(q,\epsilon)\rangle = \mathcal{N} \int d^d x \; e^{-iqt} \epsilon_{\mu\nu} T^{\mu\nu}(x)|0\rangle$$

- Mixed state $\alpha_1|\phi_i\phi_j\rangle_f + \alpha_2|T(q,\epsilon)\rangle$

Evaluating energy one-point function gives $2 \times 2$ matrix:

$$\begin{pmatrix} {}_f\langle\phi_i\phi_j|\mathcal{E}|\phi_i\phi_j\rangle_f & {}_f\langle\phi_i\phi_j|\mathcal{E}|T(q,\epsilon)\rangle \\ \langle T(q,\epsilon)|\mathcal{E}|\phi_i\phi_j\rangle_f & \langle T(q,\epsilon)|\mathcal{E}|T(q,\epsilon)\rangle \end{pmatrix} \succcurlyeq 0$$

Require this to be positive-definite $\Rightarrow$ stronger constraints

# Ongoing Work: Moving Beyond 5-point Blocks

- Apply similar methods to determine 6-point blocks in the snowflake channel for scalar 6-point functions

$$G^{(m)}_{\Delta,\ell;\Delta',\ell';\Delta'',\ell''}\Big|_{\text{snowflake}} \propto \langle\phi_{\Delta_1}(X_1)\phi_{\Delta_2}(X_2)\mathcal{O}_{\Delta,\ell}\rangle \bowtie \langle\mathcal{O}_{\Delta,\ell}\mathcal{O}_{\Delta',\ell'}\mathcal{O}_{\Delta'',\ell''}\rangle^{(m)} \bowtie \langle\mathcal{O}_{\Delta',\ell'}\phi_{\Delta_3}(X_3)\phi_{\Delta_4}(X_4)\rangle \bowtie \langle\mathcal{O}_{\Delta'',\ell''}\phi_{\Delta_5}(X_5)\phi_{\Delta_6}(X_6)\rangle$$

- Main difference: 3-point structure of type spin-spin-spin $\langle\mathcal{O}_{\Delta,\ell}\mathcal{O}_{\Delta',\ell'}\mathcal{O}_{\Delta'',\ell''}\rangle^{(m)}$

$\Rightarrow$ Consequence: Require differential operators $(\mathcal{D}^{(+0)}_{X_1}\cdot\mathcal{D}^{(-0)}_{X_6})$

$\Rightarrow$ Have two types of relations
  - **One** spin varying: with differential operators
  - **Two** spins varying: without differential operators

$\Rightarrow$ Multiple special cases

# Ongoing Work: The 5-point Conformal Bootstrap

Goal: to implement the bootstrap on the 3D critical Ising model by using results for 5-point (and later 6-point) blocks Gliozzi (2013)

- E.g. $\langle \sigma(x_1)\sigma(x_2)\epsilon(x_3)\sigma(x_4)\sigma(x_5)\rangle$ can be expanded in the (12)(45) OPE, the (14)(25) OPE, the (13)(45) OPE
- Truncate the CFT at some level $N$ by including the first $N$ conformal blocks
- Apply a numerical bootstrap method to extract CFT data
- Only works for a truncable CFT - limitation!
- Hope to obtain new OPE coefficients

# Conclusions

- Presented a concrete and practical approach to computing general symmetric traceless exchange conformal blocks appearing in 5-point functions of arbitrary scalar operators
- Derived a simple set of recursion relations using the weight-shifting formalism
- Relations allow to reduce symmetric traceless blocks to linear combinations of scalar exchange blocks with shifted dimensions

# Conclusions (cont.)

- Considered promoting one of the external operators to have spin 1 or 2
- Discussed one possible application of these results in deriving novel constraints from the ANEC in the context of 5-point functions
- Considered extending these methods to 6-point snowflake channel blocks
- Discussed ongoing efforts to implement 5-point bootstrap
- In future: May be interesting to generalize these methods to nontrivial exchanged representations

**THANK YOU!**